# Math 2010 Week 5

## Finding limit using polar coordinates

$(x,y) \longleftrightarrow (r,\theta)$ with $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

Note

$$\boxed{(x,y) \to (0,0) \iff r \to 0}$$

eg Find limits using polar coordinates

(1) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$

$$= \lim_{r\to 0} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta)$$

$$= 0 \qquad \text{(Squeeze theorem)}$$

## Details

$$|\cos^3\theta + \sin^3\theta| \le |\cos^3\theta| + |\sin^3\theta| \le 2$$

$$\therefore \ |r(\cos^3\theta + \sin^3\theta)| \le 2|r|$$

Also $\displaystyle \lim_{r\to 0} 2|r| = 0 \Rightarrow \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0$

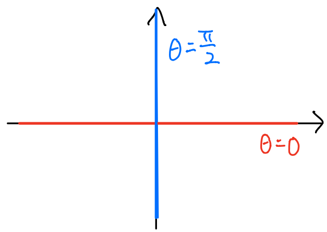
(2) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2+xy}{2(x^2+y^2)}$

$$= \lim_{r\to 0} \frac{r^2\cos^2\theta + r^2\cos\theta\sin\theta}{2r^2}$$

$$= \lim_{r\to 0} \frac{\cos^2\theta + \cos\theta\sin\theta}{2}$$

which depends on $\theta$ $\qquad \therefore$ limit DNE

eg $= \begin{cases} \frac{1}{2} & \text{if } \theta = 0 \\ 0 & \text{if } \theta = \frac{\pi}{2} \end{cases}$



③ $\lim_{(x,y)\to(0,0)} xy\ln(x^2+y^2)$

$$= \lim_{r\to 0} \underbrace{r^2\cos\theta\sin\theta\ln(r^2)}$$

Note $|\cos\theta\sin\theta| \le 1$,

$r^2 \to 0$, $\ln(r^2)\to -\infty$ as $r\to 0$

i $\left|r^2\cos\theta\sin\theta\ln(r^2)\right| \le \left|r^2\ln(r^2)\right|$

ii $\lim_{r\to 0} r^2\ln(r^2) = \lim_{r\to 0}\frac{\ln(r^2)}{\frac{1}{r^2}}$ $\left(\frac{-\infty}{\infty}\right)$

$$= \lim_{r\to 0}\frac{\frac{2r}{r^2}}{-\frac{2}{r^3}}$$ (L'Hopital's Rule)

$$= \lim_{r\to 0} -r^2 = 0$$

By Squeeze thm, $\lim_{(x,y)\to(0,0)} xy\ln(x^2+y^2) = 0$

## Iterated Limit (Au 2.7)

① $\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)$

means taking limit with $y\to 0$ first,

followed by taking limit with $x\to 0$

② Similar for $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$

Q Are they equal?

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) \overset{?}{=} \lim_{y\to 0}\lim_{x\to 0} f(x,y)$$

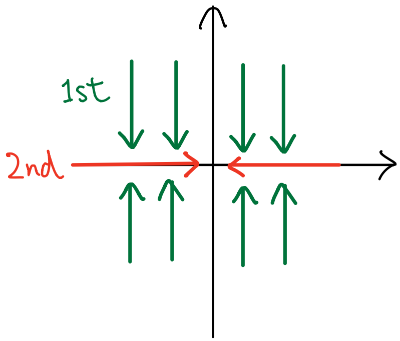
$$\overset{?}{=} \lim_{(x,y)\to(0,0)} f(x,y)$$

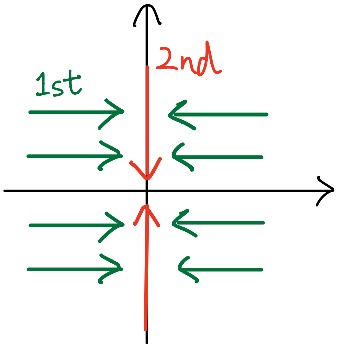
eg Consider $f(x,y) = \dfrac{x+y}{x-y}$.

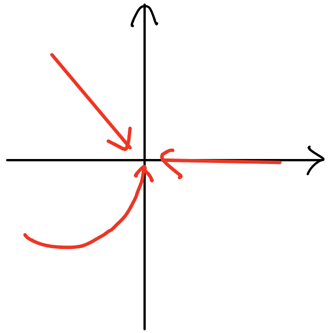
$x$ is fixed, like a constant

Not inequal

$$\lim_{x\to 0} \underbrace{\lim_{y\to 0}}\frac{x+y}{x-y} = \lim_{x\to 0}\frac{x+0}{x-0} = 1 \qquad \lim_{y\to 0}\lim_{x\to 0}\frac{x+y}{x-y} = \lim_{y\to 0}\frac{0+y}{0-y} = -1 \qquad \lim_{(x,y)\to(0,0)}\frac{x+y}{x-y} \text{ DNE}$$



Rmk

① $\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{y\to 0}\lim_{x\to 0} f(x,y)$ $\not\Longrightarrow$ $\not\Longleftarrow$ $\lim_{(x,y)\to(0,0)} f(x,y)$ exists

(Both exist and equal)

eg. $f(x,y) = \begin{cases} 1 & \text{if } x=y \\ 0 & \text{if } x\neq y \end{cases}$

eg. $f(x,y) = \begin{cases} x\cos\frac{1}{y} + y\cos\frac{1}{x} & \text{if } x,y\neq 0 \\ 0 & \text{if } x,y=0 \end{cases}$

② If all the three limits exist, they are equal

# Continuity

Let $f: A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, $\vec{a} \in A$.

**Defn 1** ($\varepsilon$-$\delta$) $f$ is said to be continuous at $\vec{a}$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $x \in A$ and $\|\vec{x} - \vec{a}\| < \delta$ then $|f(\vec{x}) - f(\vec{a})| < \varepsilon$

Another equivalent definition

**Defn 2** $f$ is said to continuous at $\vec{a}$ if $\lim\limits_{\vec{x} \to \vec{a}} f(\vec{x})$ exists and equals to $f(\vec{a})$

**Defn** $f$ is said to be continuous if $f$ is continuous at any point in $A$.

eg Let $1 \leq k \leq n$. Show $f(x_1, x_2, \cdots, x_n) = x_k$ is continuous on $\mathbb{R}^n$

Pf Let $\vec{a} = (a_1, a_2, \cdots, a_n)$.

For any $\varepsilon > 0$, take $\delta = \varepsilon$. If $\|\vec{x} - \vec{a}\| < \delta$, then

$$|f(\vec{x}) - f(\vec{a})| = |x_k - a_k| \leq \sqrt{\sum_{i=1}^{n} (x_i - a_i)^2} = \|\vec{x} - \vec{a}\| < \delta = \varepsilon$$

$\therefore \lim\limits_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a})$ for any $\vec{a} \in \mathbb{R}^n$

$\therefore f$ is a continuous function

**Thm** If $f, g: \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ are continuous at $\vec{a}$, then

① $f(x) \pm g(x)$, $kf(x)$, $f(x)g(x)$ are continuous at $\vec{a}$, where $k$ is a constant

② $\dfrac{f(\vec{x})}{g(\vec{x})}$ is continuous at $\vec{a}$ if $g(\vec{a}) \neq 0$

Pf Follow easily from corresponding properties of limit

From the last 2 results, we have

> All polynomials and rational functions (i.e. $\frac{\text{polynomial}}{\text{polynomial}}$) of multi-variables are continuous

eg ① $x^3+3yz+z^2-x+7y$ is continuous on $\mathbb{R}^3$

② $\dfrac{x^3+y^2+yz}{x^2+y^2}$ is continuous on

$$\mathbb{R}^3 \setminus \{(x,y,z): x^2+y^2=0\} = \mathbb{R}^3 \setminus \{z\text{-axis}\}$$

Let $Q(\vec{x}) = \dfrac{p_1(\vec{x})}{p_2(\vec{x})}$ be a rational function

$\vec{a}$ be a zero of $p_2(\vec{x})$ (i.e $p_2(\vec{a})=0$). Then

$Q(\vec{x})$ can be extended to a function continuous at $\vec{a}$ $\iff$ $\lim\limits_{\vec{x}\to\vec{a}} Q(\vec{x})$ exists

eg ① $f(x,y) = \dfrac{xy+y^3}{x^2+y^2}$

Note $x^2+y^2=0 \iff (x,y)=(0,0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{xy+y^3}{x^2+y^2}$$

$$= \lim_{x\to 0} \frac{mx^2+m^3x^3}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{m+m^3x}{1+m^2}$$

$$= \frac{m}{1+m^2} = \begin{cases} 0 & \text{if } m=0 \\ \frac{1}{2} & \text{if } m=1 \end{cases}$$

not equal

$\therefore$ Limit varies with slope

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ DNE

$\therefore$ $f$ cannot be extended to a function defined on $\mathbb{R}^2$

② $g(x,y)=\dfrac{x^4-y^4}{x^2+y^2}$

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$$

$$= \lim_{r\to 0} \frac{r^4\cos^4\theta - r^4\sin^4\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \lim_{r\to 0} r^2(\cos^4\theta - \sin^4\theta)$$

$$= 0 \qquad \text{(Sandwich theorem)}$$

Details: $\left|r^2(\cos^4\theta-\sin^4\theta)\right| \le 2r^2$,

and $\lim_{r\to 0} 2r^2 = 0$

$\therefore$ $g(x,y)$ can be extended to a continuous function on whole $\mathbb{R}^2$ with

$$g(x,y)=\begin{cases} \dfrac{x^4-y^4}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

**Thm** If $f:\Omega\subseteq\mathbb{R}^n\to\mathbb{R}$ is continuous at $\vec{a}$, $g(x)$ is a 1-variable function continuous at $f(\vec{a})$, then $g\circ f(\vec{x})$ is continuous at $\vec{a}$. Hence

$$\lim_{\vec{x}\to\vec{a}} g(f(\vec{x})) = g\left(\lim_{\vec{x}\to\vec{a}} f(\vec{x})\right) = g(f(\vec{a}))$$

eg $g(x)=|x|$, $f(x_1,\cdots,x_n)=x_k$ are continuous for $1\le k\le n$

$\Rightarrow$ $g\circ f(x_1,\cdots,x_n)=|x_k|$ is a continuous function

eg $g(x)=|\ln x|$ is continuous on $(0,+\infty)$

$\Rightarrow$ $g\circ f(x_1,\cdots,x_n)=|\ln x_k|$ is continuous on $\{\vec{x}\in\mathbb{R}^n : x_k>0\}$

Rmk In particular, $\lim_{(x,y)\to(0,0)} |x| = 0$, $\lim_{(x,y)\to(1,0)} |\ln x| = 0$

(We used these results in examples of sandwich thm.)

eg $\sin(x^2+yz)$, $e^{x-y}$, $\cos\left(\frac{1}{x^2+y^2}\right)$, $r=\sqrt{x^2+y^2}$

are continuous (on their domains)

# Partial Derivatives (An 3.2, Thomas 14.3)

Study the rate of change of a function with respect to each variable

**Defn** Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, $\Omega$ open. Then for $i = 1, 2, \cdots, n$, define the <u>i-th partial derivative</u> of $f$ at $\vec{x} = (x_1, \cdots, x_n) \in \Omega$

$$\frac{\partial f}{\partial x_i}(\vec{x}) = \frac{\partial f}{\partial x_i}(x_1, \cdots, x_n)$$

$$= \lim_{h \to 0} \frac{f(x_1, \cdots, x_i + h, \cdots, x_n) - f(x_1, \cdots, x_i, \cdots, x_n)}{h}$$

**eg** $\displaystyle \frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$

$\displaystyle \frac{\partial f}{\partial y}(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$

Other notations:

$$\frac{\partial f}{\partial x} = \partial_1 f = D_1 f = \nabla_1 f = f_x$$

$$\frac{\partial f}{\partial y} = \partial_2 f = D_2 f = \nabla_2 f = f_y$$

**eg** $f(x, y) = x^2 + y^2$

$\displaystyle \frac{\partial f}{\partial x} = 2x + 0 = 2x$ (Regard $y$ as a constant)

$\displaystyle \frac{\partial f}{\partial y} = 0 + 2y = 2y$ (Regard $x$ as a constant)

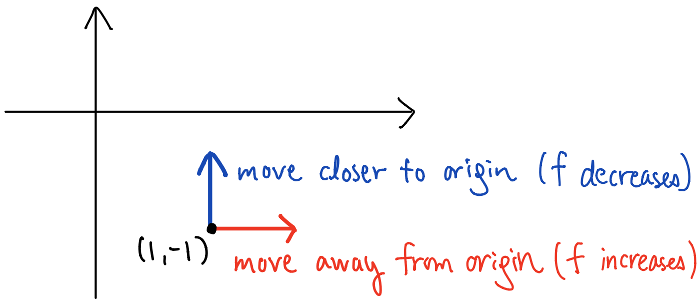
Note $\displaystyle \frac{\partial f}{\partial x}(1, -1) = 2(1) = 2 > 0$

$\displaystyle \frac{\partial f}{\partial y}(1, -1) = 2(-1) = -2 < 0$

$\therefore$ $f$ increases as $x$ increases at $(1, -1)$

decreases as $y$ increases at $(1, -1)$

**Rmk** Note $f(x, y) = \|(x, y)\|^2$



eg $f(x,y,z) = xy^2 - \cos(xz)$

Then $f_x = y^2 + z\sin(xz)$

$f_y = 2xy + 0 = 2xy$

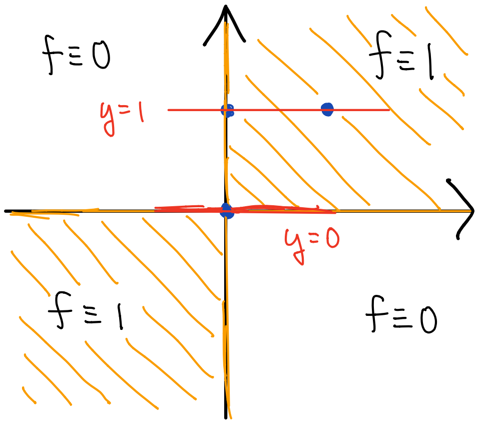
$f_z = 0 + x\sin(xz) = x\sin(xz)$

eg

$$f(x,y) = \begin{cases} 1 & \text{if } xy \geq 0 \\ 0 & \text{if } xy < 0 \end{cases}$$

Find $\frac{\partial f}{\partial x}(1,1)$, $\frac{\partial f}{\partial x}(0,1)$, $\frac{\partial f}{\partial x}(0,0)$

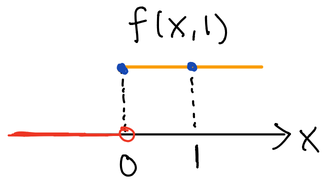
<u>Sol</u>




$\frac{\partial f}{\partial x}$ : Fix $y$, differentiate $f(x,y)$ w.r.t. $x$

<u>Along $y=1$</u>

$$f(x,1) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$




$\Rightarrow \frac{\partial f}{\partial x}(1,1) = 0$, $\frac{\partial f}{\partial x}(0,1)$ DNE

<u>Along $y=0$</u>

$f(x,0) \equiv 1$, $\forall x \in \mathbb{R}$

$\Rightarrow \frac{\partial f}{\partial x}(0,0) = 0$

<u>Rmk</u> Similarly $\frac{\partial f}{\partial y}(0,0) = 0$

Note $f$ is not continuous at $(0,0)$

$\therefore \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist at $(0,0)$ $\not\Rightarrow$ $f$ is continuous at $(0,0)$

## Higher order partial derivative

eg Consider $f(x,y)$

1st order derivative : $\frac{\partial f}{\partial x} = f_x$ , $\frac{\partial f}{\partial y} = f_y$

2nd order derivatives : $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx}$

Differentiate w.r.t. x first $\rightarrow$ $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy}$ (x first)

y first $\rightarrow$ $\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$ (y first) and $\frac{\partial^2 f}{\partial y^2} = f_{yy}$

3rd order derivatives : $\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right] = \left[\left(f_y\right)_y\right]_x = f_{yyx}$

$$\frac{\partial^3 f}{\partial x^3} = f_{xxx}$$

Others : $f_{xyy}$, $f_{yxy}$, $f_{yyy}$, $f_{xxy}$, $f_{xyx}$, $f_{yxx}$

**Rmk** Similar for higher order / more variables

eg $f_{xxyx}$ , $g_{xyz}$ for $g(x,y,z)$

eg $f(x,y) = x\sin y + y^2 e^{2x}$

Find all 1st, 2nd order partial derivatives

**Sol**

$f_x = \sin y + 2y^2 e^{2x}$

$f_y = x\cos y + 2y e^{2x}$

$f_{xx} = (f_x)_x = 4y^2 e^{2x}$

$f_{xy} = (f_x)_y = \cos y + 4y e^{2x}$

$f_{yx} = (f_y)_x = \cos y + 4y e^{2x}$

equal. Coincidence?

$f_{yy} = (f_y)_y = -x\sin y + 2e^{2x}$

**Q** Is it always true that $f_{xy}=f_{yx}$ ?

**A** No, a counter example below:

**eg** Compute $f_{xy}(0,0)$, $f_{yx}(0,0)$, where

$$f(x,y)=\begin{cases}\dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

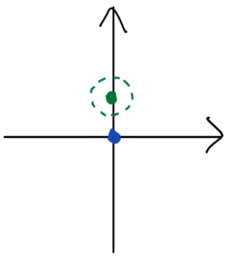
**Sol** $f_{xy}=(f_x)_y$

$$\Rightarrow f_{xy}(0,0)=\lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k}$$

Need to find $f_x(0,k)$ for $k\neq 0$ and $f_x(0,0)$

For $k\neq 0$,

$$f=\frac{xy(x^2-y^2)}{x^2+y^2} \text{ near } (0,k)$$



$$f_x=\frac{(x^2+y^2)(3x^2y-y^3)-xy(x^2-y^2)(2x)}{(x^2+y^2)^2} \text{ near } (0,k)$$

$$\Rightarrow f_x(0,k)=\frac{k^2(-k^3)-0}{k^4}=-k$$

$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$$

$$f_{xy}(0,0)=\lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k}=\lim_{k\to 0}\frac{-k-0}{k}=-1$$

Similar calculation gives $f_{yx}(0,0)=1$

(Alternatively, note $f(x,y)=-f(y,x)$

$\Rightarrow f_{yx}(0,0)=-f_{xy}(0,0)=-1$ )

In this example, $f_{xy}(0,0)\neq f_{yx}(0,0)$

**Q** When do $f_{xy} = f_{yx}$ ?

**Thm (Clairaut's Theorem / Mixed Derivative Thm)**

Let $\Omega \subseteq \mathbb{R}^2$ be an open set, $f: \Omega \to \mathbb{R}$

If $f_{xy}, f_{yx}$ exist and are continuous on $\Omega$,

then $f_{xy} = f_{yx}$ on $\Omega$

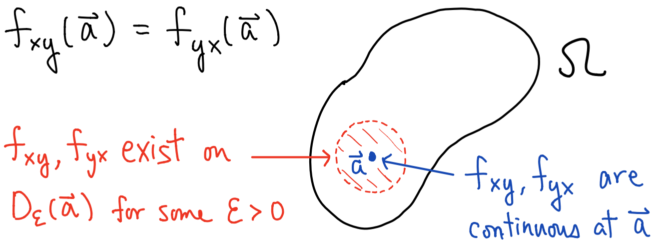
**Rmk** We will prove a stronger version:

Let $\Omega \subseteq \mathbb{R}^2$ be open, $f: \Omega \to \mathbb{R}$, $\vec{a} \in \Omega$. Suppose

① $f_{xy}, f_{yx}$ exist in an open disc containing $\vec{a}$

② $f_{xy}, f_{yx}$ are continuous at $\vec{a}$.

Then $f_{xy}(\vec{a}) = f_{yx}(\vec{a})$




Tool for the proof:

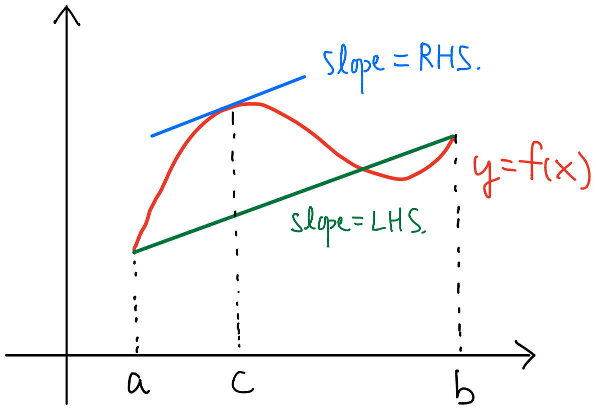
**Mean Value Theorem for 1-variable function**

Let $f: [a,b] \to \mathbb{R}$, continuous on $[a,b]$

differentiable on $(a,b)$

Then $\exists c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



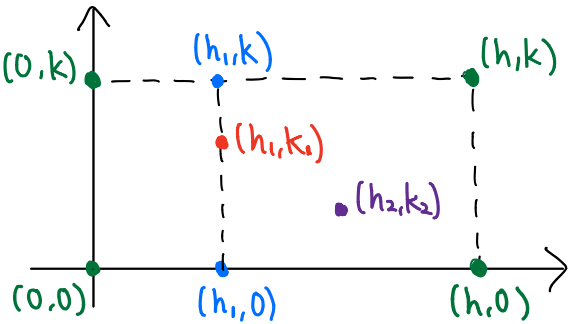
# Pf of Clairaut's Theorem

We may assume $\vec{a} = (0,0) \in \Omega$

Need to show $f_{xy}(0,0) = f_{yx}(0,0)$

Let $h, k > 0$ and $[0,h] \times [0,k] \subseteq \Omega$

$$\alpha = f(h,k) - f(0,k) - f(h,0) + f(0,0)$$




Let $g(x) = f(x,k) - f(x,0), \quad 0 \leq x \leq h$

Then $\alpha = g(h) - g(0)$

$$g'(x) = f_x(x,k) - f_x(x,0)$$

MVT $\Rightarrow \exists\, h_1 \in (0,h)$ such that

$$\frac{\alpha}{h} = \frac{g(h) - g(0)}{h} = g'(h_1) = f_x(h_1,k) - f_x(h_1,0)$$

MVT again $\Rightarrow \exists\, k_1 \in (0,k)$ such that

$$\frac{f_x(h_1,k) - f_x(h_1,0)}{k} = f_{xy}(h_1,k_1)$$

like a MVT for $f(x,y)$

$$\therefore\ \alpha = h\left[f_x(h_1,k) - f_x(h_1,0)\right] = hk\, f_{xy}(h_1,k_1)$$

Similarly, $\exists\, (h_2,k_2) \in (0,h) \times (0,k)$ such that

$$\alpha = hk\, f_{yx}(h_2,k_2)$$

$$\therefore\ f_{xy}(h_1,k_1) = f_{yx}(h_2,k_2) \quad (*)$$

Take $h,k \to 0^+$, then $(h_1,k_1), (h_2,k_2) \longrightarrow (0,0)$

$(*)$ and continuity of $f_{xy}, f_{yx}$ at $(0,0)$

$$\Rightarrow f_{xy}(0,0) = f_{yx}(0,0)$$